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APPROXIMATION AND INTERPOLATION ON COMPLEX ALGEBRAIC VARIETIES IN PSEUDOCONVEX DOMAINS

BY

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ABSTRACT

This paper investigates the twin problems of approximation and interpolation employing weighted integral representation formulas of Berndtsson–Andersson. These interpolation techniques are applied to extend local rational approximation estimates from complex algebraic complete intersection variety $\mathbb{X}$, of pure dimension n, into a strictly pseudoconvex semi-local domain D in the ambient space $\mathbb{C}^N$ with $N = n + p$, $p > 0$. We also use weighted intergral representation formulas to provide criteria for both Montessus-type convergence and convergence in logarithmic capacity of diagonal rational sequences. The logarithmic capacity we use is carefully defined via Siciak's $\mathcal{L}$-family of extremal plurisubharmonic functions.

1. Introduction

The motivation for this paper comes from studying Berndtsson's papers [Be.1, 2] and our attempt to understand Zeriahi's conditions for the existence of the best polynomial approximation on affine complex algebraic complete intersection variety $\mathbb{X}$ in [Ze.1]. In [Be.1,2], Berndtsson showed how weighted integral representation generalized the well-known Leray–Koppelman–Henkin–Ramirez integral, and specifically used the new constructs to study interpolation problems and to

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generalize Jacobi interpolation from one variable to multidimensional complex analysis.

Our main interest in this paper is to study the twin problems of approximation and interpolation using the weighted integral representation formulas introduced by Berndtsson and Andersson [Be.1,2]. We use these interpolation techniques to extend local rational approximation data from complex algebraic complete intersection variety $\mathbb{X}$ into a strictly pseudoconvex semi-local domain D in the ambient space $\mathbb{C}^N$.

The main results of this paper are Theorems 4.6. 6.1 and 6.3. Theorem 6.1, gives conditions for Montessus-type convergence while Theorems 4.6 and 6.3 give criteria for convergence in logarithmic capacity of diagonal rational sequences.

Now we turn to the description of the content of this paper. In section 2 of the paper we develop the tools needed for the construction of rational approximants to holomorphic functions in an open neighbourhood of the origin which is assumed to be in $\mathbb{X}$. In general, all the holomorphic functions on $\mathbb{X}$ are defined on the regular part $\mathbb{X}_{\text{reg}}$ of the variety $\mathbb{X}$. In section 3, we obtain an estimate in Lemma 3.4 which plays a key role in the rest of the paper. The use of this estimate is essential in the proof of the Montessus-type theorem in a semi-local domain in the complex algebraic complete intersection variety $\mathbb{X} \subset \mathbb{C}^N$. Section 4 considers diagonal rational sequences and their convergence in a logarithmic capacity to a certain class of meromorphic functions. We provide background material, sufficient for our purposes, on Siciak's $\mathcal{L}$-families of plurisubharmonic functions [Si.1,2] with logarithmic growth on non-pluripolar compact subsets K in $\mathbb{X}$. Section 5 defines the complex algebraic complete intersection variety $\mathbb{X}$ and the strictly pseudoconvex domain D in $\mathbb{C}^N$ containing $\mathbb{X}$. The conditions under which one obtains the Berndtsson's weighted interpolation intergrals are explained in section 6. This section represents an application of the constructions in section 5 to obtain the extension of rational approximation to the special class of meromorphic functions already defined.

Finally, we are extremely grateful to the referee for his careful reading of our paper, for numerous useful remarks and for suggesting corrections to some glaring errors in the original manuscript.

2. Technical definitions and notation

In this section, we let D be a domain in $\mathbb{C}^n$, which contains the origin and has a smooth C^∞ boundary ∂D. This condition will be modified in the sequel, especially from section 5 onward, to include the study of complex algebraic complete intersection varieties $\mathbb{X}$ of codimension $p > 1$ in pseudoconvex domains D in $\mathbb{C}^N$; $N := n + p$, which intersect the boundaries of the domains transversally. Let $\mathcal{O}(\overline{D})$ denote the space of functions that are holomorphic in some neighbourhood of $\overline{D}$. Denote by $\mathcal{M}er^1(\overline{D})$ the space of functions meromorphic in some neighbourhood of $\overline{D}$ satisfying the following properties:

(i) Each element of $\mathcal{M}er^1(\overline{D})$ is holomorphic at $z = 0$.

(ii) Every member of $\mathcal{M}er^1(\overline{D})$ has a polar set determined by the zeros of some nonhomogeneous polynomial on D.

Let $\mathbb{I} = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, and $\mathbb{I}^N = \mathbb{I} \times \mathbb{I} \times \cdots \times \mathbb{I}$, N-fold copy of $\mathbb{I}$. We introduce a partial ordering in $\mathbb{I}^N$ as follows: For any pair $\alpha, \beta \in \mathbb{I}^N$, $\alpha \preceq \beta \iff \alpha_j \leq \beta_j$, $j = 1, 2, \ldots, N$. $|\alpha| = \alpha_1 + \cdots + \alpha_N$, for any $\alpha = \{\alpha_1, \ldots, \alpha_N\} \in \mathbb{I}^N$.

We let $E_\mu^N := \{\alpha \in \mathbb{I}^N: 0 \leq |\alpha| \leq \mu,\ \mu \in \mathbb{I}\}$ satisfy the following conditions:

(I) $E_0^N := \{(0, \ldots, 0)\} \subset E_\mu^N$, $\forall \mu \geq 1$.

(II) $E_\lambda^N \subset E_\mu^N$ for $\lambda, \mu \in \mathbb{I} \iff 0 < \lambda < \mu$.

(III) $|E_\mu^N| = \binom{N+\mu}{N}$, where $|E_\mu^N|$ represents the cardinality of E_μ^N.

We define an index set, which will facilitate the definition of rational approximants.

Definition 2.1: A subset $E_{\mu\nu}^N$ of $\mathbb{I}^N$ is called an index interpolation set if:

(i) $E_\mu^N \subset E_{\mu\nu}^N$ for each $\mu, \nu \in \mathbb{I}$.

(ii) $\beta \in E_{\mu\nu}^N \Longrightarrow \alpha \in E_{\mu\nu}^N$, $\forall 0 \preceq \alpha \preceq \beta$.

(iii) $\exists \lambda_{\mu\nu} \in \mathbb{I}$ with $\mu + 1 \leq \lambda_{\mu\nu} \leq \mu + \nu$, $\nu > 1$, such that

$$\binom{N + \lambda_{\mu\nu} - 1}{N} \leq |E_{\mu\nu}^N| \leq \binom{N + \lambda_{\mu\nu}}{N},$$

where, $|E_{\mu\nu}^N| \leq \binom{N+\mu}{N} + \binom{N+\nu}{N} - 1$. We say that $E_{\mu\nu}^N$ is maximal if $|E_{\mu\nu}^N| \geq \binom{N+\mu}{N} + \binom{N+\nu}{N} - 1$. Now let $\mathcal{R}_{\mu\nu}$ denote the family of rational functions,

$$R_{\mu\nu} := \left\{ \frac{P_\mu(z)}{Q_\nu(z)} \right\}_{\mu,\nu},$$

where $P_\mu(z) = \sum_{k=1}^{\mu} \Phi_k(z)$ and $Q_\nu(z) = \sum_{l=1}^{\nu} \Psi_l(z)$ are nonhomogeneous polynomials of respective degrees at most μ and ν. The terms $\Phi_k(z)$ and $\Psi_l(z)$ appearing in the above expansions of P_μ and Q_ν are all homogeneous polynomials of degrees k and l respectively. Further, we require that P_μ and Q_ν be relatively prime. The relative primeness of P_μ and Q_ν holds in the whole of $\mathbb{C}^n$, except at the points of the indeterminancy of $\frac{P_\mu(z)}{Q_\nu(z)}$ in $\mathbb{C}^n$.

Definition 2.2: Let $f \in \mathcal{O}(U)$, where U is a neighbourhood of $z = 0$ such that $\mathcal{M}er^1(U) = \mathcal{O}(U)$. We say that $\frac{P_\mu(z)}{Q_\nu(z)} \in \mathcal{R}_{\mu\nu}$ is a rational approximant to f at $z = 0$ if the following conditions hold:

(1) $\partial^{|\alpha|}(Q_\nu(\xi)f(\xi) - P_\mu(\xi))|_{\xi=0} = 0$, $\forall \alpha \in E_\mu^N$.

(2) $\partial^{|\alpha|}(Q_\nu(\xi)f(\xi))|_{\xi=0} = 0$, $\forall \alpha \in E_{\mu\nu}^N \setminus E_\mu^N$.

Note that the condition (2) of Definition 2.2, together with the normalization $Q_\nu(0) = 1$ and with $E_{\mu\nu}^N$ maximal, give rise to a uniquely determined sequence $\{\frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}\}_{\mu\nu}$. To construct these rational approximants (cf. [Lu.1], [D.E-M.L.1]), we begin with the system of linear equations given by equation (2) of Definition 2.2. We use the normalization condition $Q_\nu(0) = 1$ together with maximal $E_{\mu\nu}^N$, to deduce that the maximal rank of the linear system referred to above is $\binom{N+\nu}{N} - 1$. This then yields a unique solution for the system, determining the coefficients of $Q_\nu(z)$. Once the coefficients of $Q_\nu(z)$ are obtained, we use equation (1) of Definition 2.2 to determine the coefficients of the numerator polynomial $P_\mu(z)$ of the rational approximant of f at $0 \in \mathbb{C}^n$. The resulting rational approximants are therefore uniquely determined up to the choice of $E_{\mu\nu}^N$ applied. This situation contrasts sharply with the one-variable case, where the analogous maximal $E_{\mu\nu}^1$ is itself a unique index set. In the multidimensional case, the maximal $E_{\mu\nu}^N$, for $N \geq 2$, is not a uniquely determined index set. Thus, there are as many uniquely determined unisolvent rational approximants as there are maximals $E_{\mu\nu}^N$. We note here that the above construction of rational approximants can be carried out in the most natural manner on complex algebraic complete intersection varieties $\mathbb{X} \subset \mathbb{C}^N$. In this case, the construction is done on the regular part $\mathbb{X}_{\text{reg}}$ of the complex algebraic variety $\mathbb{X}$, or, more generally, we assume in the sequel that all the zero sets of the denominator polynomials $Q_{\mu\nu}$ and that of the nonhomogeneous polynomial q, and the polar sets of the meromorphic functions involved in our constructions contain the singular set $\mathbb{X}_{\text{sing}} := \mathbb{X} \setminus \mathbb{X}_{\text{reg}}$. We observe that in the $\mathcal{L}$-capacity mode of convergence, the singular set $\mathbb{X}_{\text{sing}}$, since it is an analytic set, plays no significant role. The rational sequence, $\{\frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}\}_{\mu\nu}$, of approximants so determined is called unisolvent.

In order to deal with questions involving convergence of these rational approximants, it is often convenient to normalize the denominator $Q_{\mu\nu}(z)$ of $\frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}$ by dividing the top and bottom of $\frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}$ by the largest coefficient of $Q_{\mu\nu}(z)$. The normalized denominator polynomials, $\{\widetilde{Q_{\mu\nu}}(z)\}$, are uniformly bounded as a sequence of polynomials. Note that under the normalization process $\frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}$ remains invariant, since $\frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)} = \frac{\widetilde{P_{\mu\nu}}(z)}{\widetilde{Q_{\mu\nu}}(z)}$, where $Q_{\mu\nu}(z) \neq 0$ in U.

3. Estimates and convergence

Let $f \in \mathcal{M}er^1(\overline{D})$ and let $q(z)$ be a nonhomogeneous polynomial of degree ω which determines the polar set of f in D, i.e., $\mathcal{Z}(q) \cap D = \mathcal{Z}(f^{-1}) \cap D$, where $\mathcal{Z}(q) = \mathcal{Z}(f^{-1})$ is the zero set of the nonhomogeneous polynomial q. Then it is clear that in D, $fq \in \mathcal{O}(\overline{D})$. Next we define the function $H_{\mu\nu}(z)$ (which will play an important role in approximation as well as interpolation in this paper) by

$$H_{\mu\nu}(z) = Q_{\mu\nu}(z)f(z)q(z) - P_{\mu\nu}(z)q(z). \tag{3.1}$$

The following theorem gives the most important properties of the function $H_{\mu\nu}(z)$ for $\nu \leq \mu$.

THEOREM 3.1: *Let $f \in \mathcal{M}er^1(\overline{D})$, where the polar set of f is determined by the zero set of a nonhomogeneous polynomial q of degree ω. Suppose*

$$\pi_{\mu\nu}(z) := \frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}$$

is a (μ, ν)-rational approximant to f at $z = 0$ with its denominator polynomial $Q_{\mu\nu}(z)$ normalized. Then

(i) *$H_{\mu\nu}(z) = Q_{\mu\nu}f(z)q(z) - P_{\mu\nu}(z)q(z) \in \mathcal{O}(\overline{D})$, for $\nu \leq \mu$, and*

(ii) *$H_{\mu\nu}(z)$ converges uniformly to zero on compact subsets of D as $\mu \to \infty$.*

Before proving the theorem, we give a power series expansion of $H_{\mu\nu}(z)$ in D. Write the formal power series of $H_{\mu\nu}(z)$ in the form

$$H_{\mu\nu}(z) = \sum_{k=0}^{\infty} \lambda_{k\mu\nu\omega}(z), \tag{3.2}$$

where

$$\lambda_{k\mu\nu\omega}(z) = \frac{1}{k!} \sum_{|\alpha|=k} \left(\partial^{|\alpha|} H_{\mu\nu}(\xi)|_{\xi=0}\right) z^{\alpha} \tag{3.3}$$

is a homogeneous polynomial of degree k.

In the proof of Theorem 3.1 we use Rudin's Theorem 1.5.6 [Ru.1], which deals with the convergence of series expanded in terms of homogeneous polynomials in $\mathbb{C}^n$ appropriately adapted to our situation. The next lemma gives a precise form to the homogeneous polynomials $\lambda_{k\mu\nu\omega}(z)$.

LEMMA 3.2:

$$\lambda_{k\mu\nu\omega}(z) = \begin{cases} 0, & \mu + 1 \leq |\alpha| = k \leq \mu + \omega, \\ \frac{1}{k!} \sum_{|\alpha|=k} \left(\partial^{|\alpha|}(Q_{\mu\nu}(\xi)f(\xi)q(\xi))|_{\xi=0}\right) z^{\alpha}, & k \geq \mu + \omega + 1, \end{cases}$$

where the degree of q is ω and $\nu \leq \mu$.

Proof: First note that for $\mu + 1 \leq |\alpha| \leq \mu + \omega$, where $\omega =$ degree $q(z)$,

$$\lambda_{k\mu\nu\omega}(z) = \frac{1}{k!} \sum_{|\alpha|=k} \left(\partial^{|\alpha|}\Big(Q_{\mu\nu}(\xi)f(\xi)q(\xi) - P_{\mu\nu}(\xi)q(\xi)\Big)|_{\xi=0}\right) z^{\alpha}$$
$$= \frac{1}{k!} \sum_{|\alpha|=k} \left(\Big(\partial^{|\alpha|}(Q_{\mu\nu}(\xi)f(\xi)q(\xi)) - \partial^{|\alpha|}(P_{\mu\nu}(\xi)q(\xi))\Big)|_{\xi=0}\right) z^{\alpha}.$$

CLAIM: *For $\mu + 1 \leq |\alpha| \leq \mu + \omega$,*

$$\partial^{|\alpha|}\Big(Q_{\mu\nu}(\xi)f(\xi)q(\xi)\Big)|_{\xi=0} = \partial^{|\alpha|}\Big(P_{\mu\nu}(\xi)q(\xi)\Big)|_{\xi=0}. \tag{3.4}$$

To see this, consider the homogeneous polynomial

$$\sum_{|\alpha|=k} \left(\partial^{|\alpha|}\Big(P_{\mu\nu}(\xi)q(\xi)\Big)|_{\xi=0}\right) z^{\alpha},$$

which may be expanded according to the Leibniz rule as

$$\sum_{|\alpha|=k} \left(\partial^{|\alpha|}\Big(P_{\mu\nu}(\xi)q(\xi)\Big)|_{\xi=0}\right) z^{\alpha}$$
$$= \sum_{l=0}^{\min(\mu,k)} \binom{k}{l} \sum_{|\beta|=l} \left(\partial^{|\beta|}P_{\mu\nu}(\xi)|_{\xi=0}\right) z^{\beta} \sum_{|\gamma|=k-l} \left(\partial^{|\gamma|}q(\xi)|_{\xi=0}\right) z^{\gamma}$$
$$= \sum_{l=0}^{\mu} \binom{k}{l} \sum_{|\beta|=l} \left(\partial^{|\beta|}(Q_{\mu\nu}(\xi)f(\xi))|_{\xi=0}\right) z^{\beta} \sum_{|\gamma|=k-l} \left(\partial^{|\gamma|}q(\xi)|_{\xi=0}\right) z^{\gamma},$$

where we have used condition (1) of Definition 2.2, from which we can easily obtain

$$\partial^{|\beta|}(Q_{\mu\nu}(\xi)f(\xi))|_{\xi=0} = \partial^{|\beta|}P_{\mu\nu}(\xi)|_{\xi=0},$$

$0 \leq |\beta| = l \leq \mu$, i.e., $\beta \in E_{\mu}^{N}$. Hence

$$\sum_{|\alpha|=k} \left(\partial^{|\alpha|}\Big(P_{\mu\nu}(\xi)q(\xi)\Big)|_{\xi=0}\right) z^{\alpha} = \sum_{|\alpha|=k} \left(\partial^{|\alpha|}\Big(Q(\xi)f(\xi)q(\xi)\Big)|_{\xi=0}\right) z^{\alpha},$$

for $\mu + 1 \leq |\alpha| \leq \mu + \omega$. Thus the claim and the first part of the Lemma follow. The second part follows directly from the definition of $H_{\mu\nu}(z)$ and its formal power series. ∎

The following Lemma shows that the nonzero remaining λ's have uniform bounds in a neighbourhood Ω, of $\overline{D}$.

LEMMA 3.3: *Let $\lambda_{k\mu\nu\omega}(z)$ be as in the preceding Lemma 3.2, i.e.,*

$$\lambda_{k\mu\nu\omega}(z)=\frac{1}{k!}\sum_{|\alpha|=k}\Big(\partial^{|\alpha|}\Big(Q_{\mu\nu}(\xi)f(\xi)q(\xi)\Big)|_{\xi=0}\Big)z^{\alpha},\quad k\geq\mu+\omega+1.$$

Then $\sup_k|\lambda_{k\mu\nu\omega}|<\infty$, *for* $z\in\overline{D}$, $\nu\leq\mu$, $k\geq\mu+\omega+1$.

Proof: From the definition of $H_{\mu\nu}(z)$ in $\Omega\supset D$, where Ω is a neighbourhood of $\overline{D}$, i.e.,

$$H_{\mu\nu}(z)=Q_{\mu\nu}(z)f(z)q(z)-P_{\mu\nu}(z)q(z),$$

we see immediately that $H_{\mu\nu}(z)$ is holomorphic in $\Omega\supset D$. Thus $H_{\mu\nu}(z)$ is bounded on $\overline{D}$. Now choose $r>1$, and $M:=M(r)$ such that the polydisc $\triangle_r^n:=\{z\in D\colon \max_j|z_j|<r\}\subset\subset D$ and $1<r<\operatorname{dist}(0,\partial D)$. Then Cauchy's estimate in $\triangle_r^n$ yields

$$\Big|\partial^{|\alpha|}\Big(Q_{\mu\nu}(\xi)f(\xi)q(\xi)\Big)_{\xi=0}\Big|\leq\frac{M}{r^{|\alpha|+1}},\quad |\alpha|\geq\mu+\omega+1,$$

where $M:=\sup_{\overline{\triangle_r^n}}|H_{\mu\nu}(z)|$, so that the estimate for $|\lambda_{k\mu\nu\omega}(z)|$ with $z\in\triangle_r^n$ is given by

$$|\lambda_{k\mu\nu\omega}(z)|\leq\frac{1}{k!}\sum_{|\alpha|=k}\frac{M}{r}\Big(\frac{|z^{\alpha}|}{r^{|\alpha|}}\Big)\leq\frac{1}{k!}\sum_{|\alpha|=k}\frac{M}{\operatorname{dist}(0,\partial D)}\Big(\frac{|z^{\alpha}|}{\operatorname{dist}(0,\partial D)^{|\alpha|}}\Big),$$

as $r\to\operatorname{dist}(0,\partial D)$ and for $z\in D$.

Fix $\epsilon>0$, sufficiently small, and define $D_\epsilon:=\{z\in D\colon 0<\epsilon<\operatorname{dist}(z,\partial D)\}$ in such a way that $D_\epsilon\subset\subset D$ and $\rho_\epsilon:=\operatorname{dist}(0,\partial D_\epsilon)<\delta:=\operatorname{dist}(0,\partial D)$. Then for $z\in D_\epsilon$

$$\begin{aligned}|\lambda_{k\mu\nu\omega}(z)|&\leq\frac{1}{k!}\sum_{|\alpha|=k}\frac{M}{\delta}\Big(\frac{\rho_\epsilon}{\delta}\Big)^k\leq\frac{1}{k!}\binom{n-1+k}{n-1}\frac{M}{\delta}\Big(\frac{\rho_\epsilon}{\delta}\Big)^k\\&\leq\frac{M}{\delta}\Big(\sum_{k\geq\mu+\omega+1}\binom{n-1+k}{n-1}\Big(\frac{\rho_\epsilon}{\delta}\Big)^k\Big)\leq\frac{M}{\delta}\Big(\frac{1}{(1-\frac{\rho_\epsilon}{\delta})^{n-1}}\Big),\end{aligned}$$

where $0<\rho_\epsilon/\delta<1$. Thus $\sup_k|\lambda_{k\mu\nu\omega}(z)|<\infty$, $\forall z\in\overline{D}_\epsilon$, $k\geq\mu+\omega+1$. ∎

Proof of Theorem 3.1: From Lemma 3.3, we have $\sup_k|\lambda_{k\mu\nu\omega}(z)|<\infty$ and so by Rudin's Theorem 1.5.6. [Ru.1], we deduce that $\sum_{k\geq\mu+\omega+1}\lambda_{k\mu\nu\omega}(z)$ is compactly convergent in D_ϵ. This sum clearly goes to zero as $\mu\to\infty$ and the result follows as $\epsilon\to 0$. ∎

LEMMA 3.4: *Let $H_{\mu\nu}(z)$ be given as in Theorem 3.1. Then*

(i) $H_{\mu\nu}(z) \in \mathcal{O}(\overline{D}_\epsilon)$, *for fixed ϵ such that $0 < \epsilon < 1$,*

(ii) $\|H_{\mu\nu}(z)\|_{\overline{D}_\epsilon} \leq \frac{M}{\epsilon}\sigma_\epsilon^{\mu+\omega+1}$, *where $M = \sup_{\partial D} |H_{\mu\nu}(z)|$ and $0 < \sigma_\epsilon := \rho_\epsilon/\delta < 1$.*

Proof: (i) is immediate. We show (ii) using the estimates of the preceding Lemma. For $z \in \overline{D}_\epsilon$, we obtain

$$|H_{\mu\nu}(z)| \leq \sum_{k \geq \mu+\omega+1} |\lambda_{k\mu\nu\omega}(z)| \leq \frac{M}{\delta} \sum_{k\geq\mu+\omega+1} \frac{1}{k!}\binom{n-1+k}{n-1}{\sigma_\epsilon}^k \leq \frac{M}{\delta}{\sigma_\epsilon}^{\mu+\omega+1}.$$

Since $\frac{1}{k!}\binom{n-1+k}{n-1} < 1$, for k large

$$|H_{\mu\nu}(z)| \leq \frac{M}{\epsilon}{\sigma_\epsilon}^{\mu+\omega+1},$$

where $0 < \epsilon < \text{dist}(0, \partial D) - \text{dist}(0, \partial D_\epsilon)$. Hence $\|H_{\mu\nu}(z)\|_{\overline{D}_\epsilon} \leq \frac{M}{\epsilon}\sigma_\epsilon^{\mu+\omega+1}$. ∎

This estimate is very useful in dealing with all expressions of the function $H_{\mu\nu}$ where ν is not necessarily fixed. In fact, the estimate is valid for all $\mu \geq \nu$. We consider an application of the compact convergence of $H_{\mu\nu}$ to zero, with ν fixed at $\omega = \nu$, in order to obtain a Montessus-type convergence for rational approximants (cf. [Lu.1], [D.E-M.L.1], [de MB.1]).

THEOREM 3.5: *Let $\nu = \omega$ be fixed. Suppose $f \in \mathcal{M}er^1(\overline{D})$, with $\mathcal{Z}(f^{-1}) \cap D = \mathcal{Z}(q) \cap D$ for some nonhomogeneous polynomial q of degree ω, where $\mathcal{Z}(q) = \mathcal{Z}(f^{-1})$ is the zero set of the nonhomogeneous polynomial q. Suppose $\pi_{\mu\nu}$ is a unisolvent (μ, ν)-rational approximant to f at $z = 0$, with its denominator polynomial normalized. Then as $\mu \to \infty$, we have*

(i) $\mathcal{Z}(\pi_{\mu\nu}^{-1}) \cap \overline{D} \to \mathcal{Z}(f^{-1}) \cap \overline{D}$,

(ii) $\pi_{\mu\nu} \to f(z)$ *compactly on $D \setminus \mathcal{Z}(f^{-1})$.*

Proof: The family $\{\widetilde{Q}_{\mu\nu}(z)\}$ with fixed degree ν of the denominators of $\pi_{\mu\nu}$'s, each normalized in the sense that the maximum over all indices α of the coefficients c_α of $\widetilde{Q}_{\mu\nu}$ is set equal to one, i.e., $\max_\alpha c_\alpha = 1$, forms a uniformly bounded family in $\overline{D}$. Therefore, by Montel's theorem, there exists a subsequence $\{\widetilde{Q}_{\mu_l\nu}(z)\}$ such that $\widetilde{Q}_{\mu_l\nu}(z) \to \widetilde{Q}(z)$ uniformly on $\overline{D}$ as $l \to \infty$. Moreover, from Theorem 3.1 we get $H_{\mu\nu}(z) \to 0$, uniformly on compact subsets of D as $\mu \to \infty$, i.e., $H_{\mu\nu}(z) = Q_{\mu\nu}(z)f(z)q(z) - P_{\mu\nu}(z)q(z) \to 0$. Thus, there is an induced subsequence $\{P_{\mu_l\nu}\}$ of $\{P_{\mu\nu}\}$ which converges uniformly to $P(z)$ on $\overline{D}$ so that in the limit we obtain

$$Q(z)f(z)q(z) = P(z)q(z). \tag{3.5}$$

Since every convergent subsequence of $\{\widetilde{Q}_{\mu\nu}(z)\}$ converging satisfies equation (3.5), the full sequences $\{Q_{\mu\nu}(z)\}$ and $\{P_{\mu\nu}(z)\}$ must converge uniformly to $Q(z)$ and $P(z)$ respectively on $\overline{D}$.

Now let $\mathcal{Z}(Q)$, $\mathcal{Z}(q) = \mathcal{Z}(f^{-1})$ be the zero sets of Q and q respectively. Then one more application of equation (3.5) yields

$$\mathcal{Z}(Q) \cap \overline{D} = \mathcal{Z}(q) \cap \overline{D} = \mathcal{Z}(f^{-1}) \cap \overline{D}. \tag{3.6}$$

Hence as $\mu \to \infty$, we must have the desired result,

$$\mathcal{Z}(\pi_{\mu\nu}^{-1}) \cap \overline{D} \to \mathcal{Z}(f^{-1}) \cap \overline{D}.$$

To prove (ii) we get from a quick examination of the definition of $H_{\mu\nu}$ that if we divide by $Q_{\mu\nu}(z)q(z) \neq 0$ in D, and take the absolute value of the resulting expression, we obtain

$$\left| f(z) - \frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)} \right| = \left| \frac{H_{\mu\nu}(z)}{Q_{\mu\nu}(z)q(z)} \right|. \tag{3.7}$$

Now by (i), $\mathcal{Z}(Q_{\mu\nu}) \cap D \to \mathcal{Z}(f^{-1}) \cap D$ as $\mu \to \infty$, and so given $\xi > 0$, $\exists \mu_0$ and $\tau > 0$ such that for all $\mu > \mu_0$, $\mathcal{Z}(Q_{\mu\nu}) \cap D \subset \mathcal{N}(\mathcal{Z}(f^{-1}) \cap D, \xi)$, where $\mathcal{N}(,)$ is an ξ-neighbourhood of $\mathcal{Z}(f^{-1}) \cap D$; and so $|Q_{\mu\nu}(z)q(z)| \geq \tau$ for $z \in D \setminus \mathcal{N}(\mathcal{Z}(f^{-1}) \cap D, \xi)$. Thus from (3.7) we obtain

$$\|f(z) - \pi_{\mu\nu}(z)\|_K \leq \frac{\|H_{\mu\nu}\|_K}{\tau} \leq \frac{M}{\epsilon\tau}\sigma_\epsilon^{\mu+\nu},$$

and

$$\limsup_{\mu\to\infty} \|f(z) - \pi_{\mu\nu}(z)\|_K^{1/\mu} \leq \sigma_\epsilon < 1,$$

for any compact subset K in $D \setminus \mathcal{N}(\mathcal{Z}(f^{-1}) \cap D, \xi)$. By part (ii) of Theorem 3.1, the uniform convergence follows. Let $\xi \to 0$; then the uniform convergence extends to compact subsets K in $D \setminus \mathcal{Z}(f^{-1}) \cap D$. ∎

4. Convergence in logarithmic capacity

In this section we consider a weak-type convergence for the diagonal sequences $\{\pi_{\mu\mu}\}$ with respect to logarithmic capacity. To develop the correct concept of logarithmic capacity we borrow heavily from the theory of plurisubharmonic functions (in short, psh functions) and, in particular, the subfamily of the cone of plurisubharmonic functions studied in depth by Siciak, [Si.1,2], often known as the Siciak families of extremal plurisubharmonic functions in $\mathbb{C}^n$. These can be easily defined in a much wider sense for algebraic varieties as is done by A. Sadullaev in ([Sad.1]). However, we give below only the definition of the global families on $\mathbb{C}^n$. These families figure significantly in our study of convergence in logarithmic capacity.

Definition 4.1:

(4.1)
$$\mathcal{L} := \mathcal{L}(\mathbb{C}^n) := \{v(z)\colon v \in \mathcal{PSH}(\mathbb{C}^n); \sup\{v(z) - \log(\|z\|+1),\ z \in \mathbb{C}^n\} < \infty\},$$

where $\mathcal{PSH}(\mathbb{C}^n)$ denotes the $\mathbb{R}^+$-convex cone of plurisubharmonic functions on $\mathbb{C}^n$ with $\mathbb{R}^+ := \{x \in \mathbb{R}\colon x > 0\}$.

Definition 4.2: A set $E \subset \mathbb{C}^n$ is said to be pluripolar if for each $w \in E$ there is a neighbourhood W of w and a plurisubharmonic function ϕ defined on W such that $E \cap W \subset \{z \in W\colon \phi(z) = -\infty\}$.

Let K be any compact non-pluripolar subset of $\mathbb{C}^n$. The Siciak extremal function associated with K is defined by

$$V_K(z) := \sup\{u(z)\colon u \in \mathcal{L}; u \le 0 \text{ on } K,\ z \in \mathbb{C}^n\}. \tag{4.2}$$

Let

$$V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta) \tag{4.3}$$

be the upper semi-continuous regularization of V_K. The function V_K^* is in general not smooth on $\mathbb{C}^n \setminus K$ when $n > 1$. It is a theorem of Siciak, [Si.1], that either $V_K^* \equiv +\infty$, in which case the set K is pluripolar, or else V_K^* is plurisubharmonic and $V_K^* \in \mathcal{L}$. Also, if V_K is continuous on $\mathbb{C}^n$ we have $V_K = V_K^* \in \mathcal{L}$.

The logarithmic capacity we need is introduced via the Robin constant in the following definition.

Definition 4.3: Let $K \subset \mathbb{C}^n$ be a compact nonpluripolar set. The Robin constant $\gamma(K)$ of the compact set K is given by

$$\gamma(K) := \lim_{\|z\| \to \infty} \Big(V_K^*(z) - \log(\|z\|+1)\Big), \tag{4.4}$$

where $\|\ \|$ is the metric on the Euclidean space $\mathbb{C}^n$.

The logarithmic capacity, in short $\mathcal{L}$-Capacity, is defined by

$$\mathrm{Cap}_{\mathcal{L}}(K) := \exp(-\gamma(K)). \tag{4.5}$$

The Siciak family $\mathcal{L}$ used here to define the logarithmic capacity is closely related to the study of polynomials in n-complex variables, [Si.2]. For a compact set $K \subset \mathbb{C}^n$, we have

$$V_K(z) = \sup\left\{\frac{1}{d}\log|T(z)|: d = \text{degree}(T) \geq 1, \|T\|_K \leq 1\right\} \tag{4.6}$$

where T is a polynomial on $\mathbb{C}^n$ and $\|.\|_K$ is the norm on the $\mathbb{C}$-vector space of polynomials of degree $\leq d$, with $d \geq 1$ on $\mathbb{C}^n$. The basic property of the extremal function just defined above is contained in a theorem of Zaharyuta [Zah.1] and Siciak [Si.2]. An old result of Bremmermann then gives that any $u \in \mathcal{L}$ can be expressed as

$$u(z) = \left(\limsup_k\left(\frac{1}{k}\log|T_k(z)|\right)\right)^* := \limsup_{\zeta\to z}\left(\limsup_k\left(\frac{1}{k}\log|T_k(\zeta)|\right)\right), \tag{4.7}$$

where $\{T_k(z)\}_k$ is a sequence of polynomials of degree k which can be expanded in terms of homogeneous polynomials in $\mathbb{C}^n$.

We now turn to the discussion of weak-type convergence in $\mathcal{L}$-capacity.

Definition 4.4: A sequence of functions $\{u_j\}_{j=1}^{\infty}$ is said to converge to a function u in $\mathcal{L}$-capacity, $\mathrm{Cap}_{\mathcal{L}}$, on a set $E \subset \mathbb{C}^n$, if for each $\delta > 0$ we have

$$\mathrm{Cap}_{\mathcal{L}}\left(\{\zeta \in E: |u_j(\zeta) - u(\zeta)| > \delta\}\right) \to 0, \quad \text{as } j \to \infty.$$

Recall the definition of the function $H_{\nu\nu}(z)$ (cf. Theorem 3.1) from which we deduce for $Q_{\nu\nu}(z)q(z) \neq 0$ the expression

$$f(z) - \pi_{\nu\nu}(z) = \frac{H_{\nu\nu}(z)}{Q_{\nu\nu}(z)q(z)}. \tag{4.8}$$

Remark 4.5: There are other equivalent concepts of logarithmic capacity; one interesting version was given by Pleśniak, [Pl.1]. Fix a compact subset K in $\mathbb{C}^n$. Then for any set $E \subset \mathbb{C}^n$, define its capacity as follows:

$$\alpha(E) := \alpha_K(E) := \exp\left(-\|V_E^*\|_K\right).$$

Note that $\alpha \geq 0$, and, in particular, if $E \subset F \subset \mathbb{C}^n$, then $\alpha(E) \leq \alpha(F)$.

Let $\eta > 0$ be given. Let K be a compact nonpluripolar polynomially convex subset of $\mathbb{C}^n$. Set $X := K \cap \overline{D}$. Define

$$\Omega_{\eta,\mu} := \{z \in X: |f(z) - \pi_{\mu\mu}(z)|^{1/\mu} > 1/\eta\}.$$

We shall write $T_{\mu+\omega}(z) := Q_{\mu\mu}(z)q(z)$ as a polynomial of degree at most $\mu + \omega$ in $\mathbb{C}^n$. We can now state the main theorem of this section.

THEOREM 4.6: *Let $f \in \mathcal{M}er^1(\overline{D})$ with its polar set satisfying $\mathcal{Z}(f^{-1}) \cap D = \mathcal{Z}(q) \cap D$, where q is some nonhomogeneous polynomial in $\mathbb{C}^n$ of fixed degree at most ω which determines the polar set of f in D. Suppose $\{\pi_{\mu\mu}\}_\mu$ is a (μ, μ)-diagonal sequence of unisolvent rational approximants to f at $z = 0$. Then for $\eta \in]0, 1[$ and $X = K \cap \overline{D}$, the logarithmic capacity of $\Omega_{\eta,\mu}$ satisfies the inequality*

$$Cap_{\mathcal{L}}(\Omega_{\eta,\mu}) \leq C\eta,$$

for μ sufficiently large and for some positive constant $C > 1$.

Remark 4.7: In calculations with the logarithmic capacity $\mathrm{Cap}_{\mathcal{L}}$ in $\mathbb{C}^n$, it is often expedient to compute the Pleśnaik capacity α in $\mathbb{C}^n$. The latter capacity is equivalent to the logarithmic capacity as seen from the following proposition.

PROPOSITION 4.8 ([Tay.1], [L-T.1]): *Let E be a compact nonpluripolar subset of $\mathbb{C}^n$. Let V_E^* be the upper semi-continuous regularization of*

$$V_E(z) := \sup\{v(z) : v \in \mathcal{L} := \mathcal{L}(\mathbb{C}^n); v \leq 0, \text{ on } E, z \in \mathbb{C}^n\},$$

with $V_E^* \in \mathcal{L}$. Then for some $M > 0$, $\delta \in]0, 1[$, the Pleśnaik capacity $\alpha(E) = \exp(-\|V_E^*\|)$ is equivalent to the logarithmic capacity $Cap_{\mathcal{L}}(E)$. The equivalence is given by the B. A. Taylor estimates ([Tay.1]):

$$\alpha(E) \leq Cap_{\mathcal{L}}(E) \leq M\Big(\alpha(E)\Big)^{\delta}, \tag{4.9}$$

where $Cap_{\mathcal{L}}(E) = \exp\Big(-\gamma(E)\Big)$ and $\gamma(E)$ is the Robin constant. ∎

Proof of Theorem 4.6: Let $T_{\mu+\omega}(z) = Q_{\mu\mu}(z)q(z)$, so that $T_{\mu+\omega}(z)$ is a polynomial of degree $\mu + \omega$ expandable in terms of homogeneous polynomials in $\mathbb{C}^n$. For $z \in \Omega_{\eta,\mu}$ and $\epsilon > 0$ as in Lemma 3.4, we have

$$|T_{\mu+\omega}(z)| < \frac{\sigma_\epsilon M}{\epsilon\eta^\omega}(\sigma_\epsilon\eta)^{\mu+\omega},$$

so that

$$\limsup_{\mu+\omega\to\infty} |T_{\mu+\omega}(z)|^{\frac{1}{\mu+\omega}} \leq \sigma_\epsilon\eta < \eta.$$

In the above estimates, the η used was arbitrary. Hence we can make a suitable choice of η so that $\|T_{\mu+\omega}(z)\|_X \leq 1$ on compacts. Now using (4.6), for the polynomial $T_{\mu+\omega}(z)$ with $\deg T_{\mu+\omega} \geq 1$ on $\mathbb{C}^n$, we set

$$V_X(z) = \sup\Big\{\frac{1}{\mu+\omega}\log|T_{\mu+\omega}(z)|\Big\}.$$

We see that the upper-semicontinuous regularization of $V_X(z)$, which is given by $V_X^*(z) = \limsup_{\zeta\to z} V_X(\zeta)$, belongs to the Siciak space of plurisubharmonic functions $\mathcal{L} := \mathcal{L}(\mathbb{C}^n)$. This allows us to define the level set of the extremal function by

$$\Gamma_{\eta,\epsilon} := \{z \in X: V_X^*(z) < \log(\eta+\epsilon)\}.$$

Then

$$\alpha(\Gamma_{\eta,\epsilon}) = \exp\Big(-\|V_{\Gamma_{\eta,\epsilon}}\|_X\Big) < \eta+\epsilon.$$

Moreover, for μ sufficiently large, $\Omega_{\eta,\mu} \subset \Gamma_{\eta,\epsilon}$ so that

$$\alpha(\Omega_{\eta,\mu}) \leq \alpha(\Gamma_{\eta,\epsilon}) < \eta+\epsilon.$$

Let $\epsilon \to 0$; then we get $\alpha(\Omega_{\eta,\mu}) \leq \eta$. From the proposition there exists $C > 1$ such that $\mathrm{Cap}_{\mathcal{L}}(\Omega_{\eta,\mu}) < C\eta$. ∎

5. Rational interpolation from varieties in pseudoconvex domains

In this section we bring together the constructions in the previous sections and use the techniques of weighted integral representations of holomorphic functions introduced by M. Andersson and B. Berndtsson in [B-A.1] to interpolate rational functions from complex algebraic complete intersection varieties in pseudoconvex domains with special properties. The weighted holomorphic integral representation formulae generalize those previously constructed by Leray–Henkin–Ramirez–Koppelman. B. Berndtsson, [Be.1], has successfully applied weighted integral kernels to the study of the interpolation problem.

Let D be a strictly pseudoconvex domain in $\mathbb{C}^N$ given by a C^2 defining function $\rho\colon \mathbb{C}^N \to [-\infty,\infty[$ as follows:

$$D := \{z \in \mathbb{C}^N: \rho(z) < 0\}$$

and $0 \in D$. Let $\mathbb{X}$ be a complex algebraic variety in D defined by

$$\mathbb{X} := \{\zeta \in D: \phi_1(\zeta) = \cdots = \phi_p(\zeta) = 0\},$$

where ϕ_ν, $\nu = 1,2,\ldots,p$ are polynomials on $\mathbb{C}^N$. We endow $\mathbb{X}$ with the metric and complex topology induced from the Euclidean space $\mathbb{C}^N$. $\mathbb{X}$ satisfies the complete intersection condition

$$\partial\phi := \partial\phi_1 \wedge \partial\phi_2 \wedge \cdots \wedge \partial\phi_p \neq 0, \tag{5.0}$$

with $\phi := \{\phi_1,\ldots,\phi_p\}$. In addition, we impose a transversality condition on the complex algebraic complete intersection varieties $\mathbb{X}$ we consider, and the condition on $\mathbb{X} \cap \partial D$ is given by $\partial\phi \wedge \partial\rho \neq 0$. Let Σ be an open relatively compact subset of D with $0 \in \Sigma$. Set $\mathbb{X}_\Sigma := \mathbb{X} \cap \Sigma$ such that $\mathbb{X}_\Sigma$ is a homogeneous subvariety of the complex algebraic complete intersection variety $\mathbb{X}$ with the origin $0 \in \mathbb{X}_\Sigma$, where here, homogeneous varieties in $\mathbb{C}^N$ are varieties Y in $\mathbb{C}^N$ which satisfy the condition that $y \in Y$ if and only if $\lambda y \in Y$, $\forall \lambda \in \mathbb{C}$ (cf. [M-T-V.1]). Denote by $\mathcal{O}(\overline{\mathbb{X}}_\Sigma)$ the space of holomorphic functions in a neighbourhood of $\overline{\mathbb{X}}_\Sigma$. Let $\mathcal{M}er(\overline{\mathbb{X}}_\Sigma)$ be the space of functions meromorphic in some neighbourhood of $\overline{\mathbb{X}}_\Sigma$. As before, we consider the special subclass $\mathcal{M}er^1(\overline{\mathbb{X}}_\Sigma)$ of meromorphic functions in $\mathcal{M}er(\overline{\mathbb{X}}_\Sigma)$ satisfying the properties:

P_1: $f \in \mathcal{M}er^1(\overline{\mathbb{X}}_\Sigma)$ if f is holomorphic in some neighbourhood of the origin $0 \in \mathbb{X}_\Sigma$.

P_2: $f \in \mathcal{M}er^1(\overline{\mathbb{X}}_\Sigma)$ if its polar set in $\mathbb{X}_\Sigma$ is determined by the zero set of the restriction to $\mathbb{X}_\Sigma$ of a nonhomogeneous polynomial q in $\mathbb{C}^N$. This means that to each $f \in \mathcal{M}er^1(\overline{\mathbb{X}}_\Sigma)$, there exists a nonhomogeneous polynomial q in $\mathbb{C}^N$ such that $\mathcal{Z}(q\lceil\mathbb{X}_\Sigma) = \mathcal{Z}(f^{-1})$ in $\mathbb{X}_\Sigma$, where the symbol $\lceil$ stands for restriction to the appropriate set.

We collect here some definitions from the work of Andersson and Berndtsson concerning weighted integral representations of holomorphic functions that will be needed in our work. For complete details we refer the reader to their original paper, [B-A.1]. We introduce on a strictly pseudoconvex domain $D \subset \mathbb{C}^N$ the Leray map Φ: $\overline{D} \times \overline{D} \to \mathbb{C}^N$: $(\zeta, z) \mapsto \Phi(\zeta, z)$. $\Phi(\zeta, z)$ is holomorphic in the z variable and satisfies the following properties:

$$\Phi(\zeta, z) = \sum_j \lambda_j(\zeta, z)(z_j - \zeta_j), \tag{5.1}$$

where $\lambda_j(\zeta, z) \in C^1(\overline{D} \times \overline{D})$ and is holomorphic in z;

$$\exists \delta > 0: \quad \Re\Phi(\zeta, z) \le \rho(z) - \rho(\zeta) - \delta\|z - \zeta\|^2, \tag{5.2}$$

with $\|\ \ \|$ the metric on the Euclidean space $\mathbb{C}^N$ and $\Re\Phi$ denotes the real part of Φ. We define a 1-form Ψ which is used in the construction of the Andersson–Berndtsson weighted kernels by

$$\Psi(\zeta, z) := \frac{\sum_j \lambda_j(\zeta, z) d\zeta_j}{\rho}, \tag{5.3}$$

where ρ is the defining function for the strictly pseudoconvex domain $D \subset \mathbb{C}^N$. Let $G(\alpha) := \alpha^{-m}$, $m > 0$ with $G \in C^\infty(\mathbb{C})$ and $G(1) = 1$.

B. Berndtsson's interpretation of their holomorphic representation kernel with weights in the context of the interpolation problem in complex algebraic complete intersection varieties, [Be.1], is central to our problem. We consider here only the portion of their representation kernel relevant to interpolation of rational functions from a complex algebraic complete intersection variety $\mathbb{X}$ in a strictly pseudoconvex domain D to the entire domain D. Let $\mathbb{X}$ be the complex algebraic variety satisfying condition (5.0) together with the transversality condition. Define the weighted kernel by

$$P(\zeta, z) := \frac{(N-1)!}{p!(N-p)!} G^{(N-1)}\Big(\langle \Psi, z-\zeta\rangle + 1\Big)(\overline{\partial}\Psi)^{N-p} \wedge \mu. \tag{5.4}$$

Here, $G^{(k)}$ stands for the k^{th} derivative of G, and μ is the (p,p) positive current in ζ on $\mathbb{X}$ with measure coefficients defined by

$$\mu := C_P \frac{g^1 \wedge \cdots \wedge g^p \wedge \overline{\partial}\phi_1 \wedge \cdots \wedge \overline{\partial}\phi_p}{\|\partial\phi\|^2} d\mathbb{X}_p. \tag{5.5}$$

$d\mathbb{X}_p$ represents the surface measure on $\mathbb{X}$ and depends only on the z-variable in D. Here, $\partial\phi = \partial\phi_1 \wedge \cdots \wedge \partial\phi_p \neq 0$ and the g^k, $k = 1, 2, \ldots, p$ are obtained from

$$g^k := \sum_j^p g_j^k d\zeta_j, \tag{5.6}$$

where $g_j^k \in C^1(\mathbb{X} \cap \overline{D})$. The g_j^k are in fact determined in Berndtsson, [Be.1], from the Hefer formula

$$\phi_j(z) - \phi_j(\zeta) = \sum_k^p g_j^k (z_k - \zeta_k). \tag{5.6}$$

PROPOSITION 5.1: *The kernel $P(\zeta, z)$ has the form*

$$P(\zeta, z) = C_{n,p} \frac{\rho^{N-p+m}}{\Big(\Phi(\zeta, z) + \rho\Big)^{N-p+m}} (\overline{\partial}\Psi)^{N-p} \wedge \mu. \tag{5.7}$$

Proof: Compute the $G^{N-p}(\alpha)$, i.e., the $(N-p)^{\text{th}}$ derivative of α^{-m}, and make use of (5.1) and (5.3). ∎

We now state B. Berndtsson's theorem on interpolation of holomorphic functions from a complex algebraic complete intersection variety $\mathbb{X}$ in a domain D to the domain itself. We will apply this result to interpolation of rational functions.

THEOREM 5.2 ([Be.1]): *Let h be a holomorphic function on a complex algebraic complete intersection variety $\mathbb{X}$ with $h \in C^1(\overline{\mathbb{X}})$. Then the formula*

$$H(z) = C_{n,p} \int_{\mathbb{X}} h(\zeta) P(\zeta, z), \tag{5.8}$$

defines a holomorphic function in the domain D, so that $H\lceil\mathbb{X} = h$. ∎

We use the above theorem to obtain results on rational interpolation and approximation on a special class of meromorphic functions.

THEOREM 5.3: *Let* $f \in \mathcal{M}er^1(\overline{\mathbb{X}}_\Sigma)$, *that is, there exists a nonhomogeneous polynomial* q *in* $\mathbb{C}^N$ *such that its restriction to* $\mathbb{X}_\Sigma$ *gives* $qf \in \mathcal{O}(\overline{\mathbb{X}}_\Sigma)$. *Let* $\pi_{\mu\nu}(z) := \frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)}$ *be a unisolvent rational approximant to* f *at* $z = 0 \in \mathbb{X}_\Sigma$. *Then*

(i) $h_{\mu\nu} := Q_{\mu\nu}qf - qP_{\mu\nu} \in \mathcal{O}(\overline{\mathbb{X}}_\Sigma)$,

(ii) $h_{\mu\nu}$ *has a holomorphic extension* $H_{\mu\nu} \in \mathcal{O}(\Omega)$ *such that* $H_{\mu\nu}\lceil\mathbb{X}_\Sigma = h_{\mu\nu}$.

Proof: (i) is immediate and (ii) follows from Theorem 5.2 with

$$H_{\mu\nu}(z) = C_{n,p}\int_{\mathbb{X}_\Sigma} h_{\mu\nu}(\zeta)P(\zeta, z). \quad \blacksquare \tag{5.9}$$

It remains to obtain estimates for the integral (5.9). First, a lemma of Berndtsson, given below, estimates the kernel $P(\zeta, z)$.

LEMMA 5.4 ([Be.1]): *Let* $P := P(\zeta, z)$ *be the kernel defined in (5.4). Then there exists a constant* $\kappa >$ *such that*

$$\|P\| \leq \kappa|\rho|^{m-1}\|\phi\|^{-2p}. \quad \blacksquare \tag{5.10}$$

We now use Lemma 5.4 to obtain

LEMMA 5.5: *Let* $\epsilon > 0$ *and define*

$$\Omega_\epsilon := \{z \in \Omega\colon 0 < \epsilon < \operatorname{dist}(z, \partial\Omega)\},$$

so that $\Omega_\epsilon \subset \Omega$. *Suppose the hypothesis of Theorem 5.3 holds. Then we have*

$$\|H_{\mu\nu}(z)\|_{\overline{\Omega}_\epsilon} \leq C_1\|h_{\mu\nu}(\zeta)\|_{\overline{\mathbb{X}}_\Sigma}. \tag{5.11}$$

Proof: From (5.9) and for $z \in \overline{\Omega}_\epsilon$ we have

$$\begin{aligned}\|H_{\mu\nu}(z)\| &\leq |C_{n,p}|\int_{\mathbb{X}_\Sigma} |h_{\mu\nu}(\zeta)||P(\zeta, z)| \\ &\leq \|h_{\mu\nu}(\zeta)\|_{\overline{\mathbb{X}}_\Sigma}|C_{n,p}|\int_\Omega |\rho|^{m-1}\|\phi\|^{-2p}d\mathbb{X}_p.\end{aligned}$$

The integral $\int_\Omega |\rho|^{m-1}(\|\phi\|+\epsilon)^{-2p}d\mathbb{X}_p < \infty$ is bounded (cf. [Be.1]), before letting $\epsilon \to 0$ to give its required value. We set C_1 to be the constant arising from the integral and $|C_{n,p}|$. Then the desired result is immediate,

$$\|H_{\mu\nu}(z)\|_{\overline{\Omega}_\epsilon} = \sup_{\overline{\Omega}_\epsilon}|H_{\mu\nu}| \leq C_1\|h_{\mu\nu}(\zeta)\|_{\overline{\mathbb{X}}_\Sigma}. \quad \blacksquare$$

6. Division and convergence of rational interpolants

In the preceding section we have used Berndtsson's weighted integral kernels [Be.1] to interpolate holomorphic functions from a homogeneous subvariety $\mathbb{X}_\Sigma$ in a complex algebraic complete intersection variety $\mathbb{X}$ contained in D to the whole strictly pseudoconvex domain D. In this final section our concern is with rational approximation, and for this the following formula is central to our results:

$$F(z) - \frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)} = \frac{H_{\mu\nu}(z)}{Q_{\mu\nu}(z)q(z)}, \tag{6.1}$$

for $z \in D \setminus \{\zeta \in \mathbb{C}^N\colon Q_{\mu\nu}(\zeta)q(\zeta) = 0\}$, where $Q_{\mu\nu}q$ is a polynomial on $\mathbb{C}^N$ of degree less than or equal to $\nu + \omega$.

Next, we discuss the extension of the modes of convergence described in sections 3 and 4, in particular, the convergence phenomena in the statements of Theorems 3.5 and 4.6. In order not to be unduly repetitive in the rest of our discussions, we shall refer to the convergence phenomena in Theorems 3.5 and 4.6 as Montessus-type convergence and convergence in logarithmic capacity respectively.

THEOREM 6.1: *Let $\nu \in \mathbb{I}$ be fixed. Suppose $F \in \mathcal{M}er^1(\overline{D})$ and let q be a nonhomogeneous polynomial on $\mathbb{C}^N$ of degree ν such that*

$$\mathcal{Z}(F^{-1}) \cap D = \mathcal{Z}(q) \cap D.$$

Suppose, further, that the restriction $F\lceil\mathbb{X}_\Sigma = f$ is meromorphic in $\mathbb{X}_\Sigma$ satisfying conditions P_1 and P_2 in section 5; and also:

(a) *$\Pi_{\mu\nu}\lceil\mathbb{X}_\Sigma$ is a unisolvent rational approximant to f at $0 \in \mathbb{X}_\Sigma$.*

(b) *$\Pi_{\mu\nu}\lceil\mathbb{X}_\Sigma$ has a Montessus-type convergence to f in $\mathbb{X}_\Sigma$.*

Then $\Pi_{\mu\nu}$ has an extension Montessus-type convergence to F in Σ.

We note here that in the Montessus-type convergence the degrees of $Q_{\mu\nu}$ and q are the same.

LEMMA 6.2: *Let $Q_{\mu\nu}$ and q be nonhomogeneous polynomials in $\mathbb{C}^N$ of the same fixed degree ν with $\mu \geq \nu$. Let $Q_{\mu\nu}\lceil\mathbb{X}_\Sigma$ and $q\lceil\mathbb{X}_\Sigma$ be the restrictions of $Q_{\mu\nu}$ and q to $\mathbb{X}_\Sigma$. Suppose $Q_{\mu\nu}\lceil\mathbb{X}_\Sigma \to q\lceil\mathbb{X}_\Sigma$ uniformly in $\mathbb{X}_\Sigma$. Then $Q_{\mu\nu} \to q$ uniformly in $\overline{\mathbb{X}}_\Sigma$.*

Proof: This follows from Berndtsson's weighted integral formula and the uniform convergence of the sequence $\{Q_{\mu\nu}\lceil\mathbb{X}_\Sigma\}_\mu$ to $q\lceil\mathbb{X}_\Sigma$ as $\mu \to \infty$. Since we know that

$Q_{\mu\nu}\lceil\mathbb{X}_\Sigma$ and $q\lceil\mathbb{X}_\Sigma$ are holomorphic, then as $\mu \to \infty$, we get from (5.8) that

$$\begin{array}{ccc} Q_{\mu\nu}(z) & = & C_{n,p}\int_{\overline{\mathbb{X}}_\Sigma}(Q_{\mu\nu}\lceil\mathbb{X}_\Sigma)(\zeta)P(\zeta,z) \\ \downarrow & & \downarrow \\ q(z) & = & C_{n,p}\int_{\overline{\mathbb{X}}_\Sigma}(q\lceil\mathbb{X}_\Sigma)(\zeta)P(\zeta,z) \end{array} \qquad \blacksquare$$

Proof of Theorem 6.1: First we choose a compact subset K of Σ so that

$$K \subset \Sigma_\epsilon \setminus \{\zeta \in \mathbb{C}^N : Q_{\mu\nu}(\zeta)q(\zeta) = 0\},$$

for μ sufficiently large and $\epsilon > 0$, where $\Sigma_\epsilon = \{z \in \Sigma : 0 < \epsilon < \operatorname{dist}(z, \partial\Sigma)\}$. Then for $z \in K$ we obtain from (6.1) that

$$\left\| F(z) - \frac{P_{\mu\nu}(z)}{Q_{\mu\nu}(z)} \right\|_K \leq \frac{\|H_{\mu\nu}(z)\|_K}{\inf_K |Q_{\mu\nu}(z)q(z)|}. \tag{6.2}$$

From Lemma 6.2, we know that for sufficiently large μ, there exists $\beta > 0$ such that $\inf_K |Q_{\mu\nu}(z)q(z)| \geq \beta > 0$. Now combining this with Lemmas 3.4 and 5.5 together with inequality (5.11) we get

$$\limsup_{\mu\to\infty} \|F(z) - \Pi_{\mu\nu}(z)\|_K^{1/\mu} \leq \sigma_\epsilon < 1,$$

where $\Pi_{\mu\nu}\lceil\mathbb{X}_\Sigma = \pi_{\mu\nu}$, for

$$K \subset \Sigma_\epsilon \setminus \{\zeta \in \Sigma : q(\zeta) = 0\} = \Sigma_\epsilon \setminus \mathcal{Z}(F^{-1}),$$

since by Lemma 6.2, $Q_{\mu\nu}(z) \to q(z)$ uniformly on $\overline{\Sigma}$. Thus we obtain

$$\limsup_{\mu\to\infty} \|F(z) - \Pi_{\mu\nu}(z)\|_K^{1/\mu} \leq \sigma_\epsilon < 1,$$

where $K \subset \Sigma \setminus \mathcal{Z}(F^{-1})$, since $\Sigma_\epsilon \subset \Sigma$; note that the rate of convergence is clearly geometric. Finally, observe that Lemma 6.2 also shows that

$$\mathcal{Z}(\Pi_{\mu\nu}^{-1}) \cap \Sigma \to \mathcal{Z}(F^{-1}) \cap \Sigma,$$

thus establishing the desired result. ∎

Next we consider the analog of the convergence in logarithmic capacity for the diagonal sequences $\{\Pi_{\mu\mu}\}_\mu$ of unisolvent rational approximants.

THEOREM 6.3: *Let F and its restriction $F\lceil\mathbb{X} = f$ satisfy the same conditions as in Theorem 6.1. Suppose $\Pi_{\mu\mu}\lceil\mathbb{X}_\Sigma$ is a unisolvent diagonal rational approximant to f at $0 \in \mathbb{X}_\Sigma$. Then if $\Pi_{\mu\mu}\lceil\mathbb{X}_\Sigma$ converges in logarithmic capacity to f in $\mathbb{X}_\Sigma$, so likewise $\Pi_{\mu\mu}$ converges in logarithmic capacity to F in Σ.*

Proof: From Theorem 5.3, we know that

$$H_{\mu\mu}(z) = C_{n,p} \int_{\overline{\mathbb{X}}_\Sigma} h_{\mu\mu}(\zeta)P(z,\zeta),$$

for $z \in \Sigma$ a holomorphic extension of $h_{\mu\mu}$ in $\mathbb{X}_\Sigma$, with $P(z,\zeta)$ the Berndtsson–Andersson weighted kernel.

Now since $Q_{\mu\mu}(\zeta)q(\zeta)$ is holomorphic because it is a polynomial in $\mathbb{X}_\Sigma$, it has an extension, which we denote by $Q^{Ext}_{\mu\mu}(z)q^{Ext}(z)$ in Σ, by Theorem 5.3. We use the estimate from Lemmas 3.4 and 5.5 to get

$$\|H_{\mu\mu}(z)\|_{\overline{\Sigma}_\epsilon} \leq C_1\|h_{\mu\mu}(\zeta)\|_{\overline{\mathbb{X}}_\Sigma} \leq C_1\frac{M}{\epsilon}\sigma^{\mu+\omega+1}.$$

Here, $\Sigma_\epsilon := \{z \in \Sigma\colon 0 < \epsilon < \operatorname{dist}(z,\partial\Sigma)\}$.

Let $\mathcal{Y}_\mu := \{z \in \mathbb{C}^N\colon Q^{Ext}_{\mu\mu}(z)q^{Ext}(z) = 0\}$. Then for $z \in \Sigma_\epsilon \setminus \mathcal{Y}_\mu$, we have, recalling that $\Pi_{\mu\mu}\lceil V_\Sigma = \pi_{\mu\mu}$,

$$\|F(z) - \Pi_{\mu\mu}(z)\|_{\overline{\Sigma_\epsilon \setminus \mathcal{Y}_\mu}} \leq C_1\frac{M\sigma^{\mu+\omega+1}}{\epsilon|Q^{Ext}_{\mu\mu}q^{Ext}(z)|}.$$

Let θ be given satisfying $0 < \theta < 1$. If we set

$$\Omega^{Ext}_{\theta,\mu} := \{z \in \Sigma_\epsilon\colon |F(z) - \Pi_{\mu\mu}(z)|^{1/\mu} > 1/\theta\},$$

and write $T^{Ext}_{\mu+\omega}(z) := Q^{Ext}_{\mu\mu}(z)q^{Ext}(z)$, a polynomial of degree $\mu+\omega$ in $\mathbb{C}^N$, then we obtain

$$|T^{Ext}_{\mu+\omega}(z)| < \frac{\sigma_\epsilon M(\theta\sigma_\epsilon)^{\mu+\omega}}{\epsilon\theta^\omega}.$$

Following the same line of reasoning as in the proof of Theorem 4.6, we conclude that there exists a constant $C' > 0$ such that

$$\operatorname{Cap}_{\mathcal{L}}(\Omega^{Ext}_{\theta,\mu}) < C'\theta. \quad \blacksquare$$

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